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Stochastic Stability: A Review and Some Perspectives

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Abstract A review of the stochastic stability property for the Gaussian spin glass models is presented and some perspectives discussed.

Keywords Stochastic stability · Spin glass identities

1 Introduction

In this paper we review the property of *stochastic stability* originally introduced for the Sherrington-Kirkpatrick spin-glass mean field model in [1]. Here we show some of its consequences expressed in terms of the quenched equilibrium state both in the form of identities for the overlap distribution and of quenched additivity of the free energy. Overlaps between two spin configurations $\sigma^{(1)}$ and $\sigma^{(2)}$ are usually defined as

$$q(\sigma^{(1)},\sigma^{(2)}) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^{(1)}\sigma_i^{(2)}, \tag{1}$$

but our treatment is given in full generality since it is by now well known [2] that every Gaussian spin glass model has an equilibrium state well expressed by the properties of a probability measure of a suitable overlap structure given by its covariance matrix. Stochastic stability provided a first and simple method to produce an infinite family of identities for the overlap variables. Identities for random variables with respect to the quenched state reduce the degrees of freedom of the model and go toward the core of the Parisi mean-field theory: spin glasses are described by a probability distribution of a single overlap variable and the collections of copies necessary to describe the whole equilibrium state can be obtained by a suitable combinatorial rule called *ultrametric* which holds for classes of equivalent overlap structures (*overlap equivalence*). Although a similar research project is not even completed for the Sherrington-Kirkpatrick model important progresses have been done toward it and

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there are clear indications, some based on numerical work [3–5] some on rigorous grounds (see the last section), that mean field models and short-range finite-dimensional ones behave quite similarly as far as the factorization rules are concerned. Stochastic stability is deeply rooted also within the physics community. It has immediately been used in fact in [6, 7] to determine a relation between the off-equilibrium dynamics which is experimentally accessible and the static properties and is considered, from the theoretical point of view, a structural property of the spin glass phase [8, 9].

The paper is organized as follows: the first section introduce the basic notions of a spin glass systems and the relative notations, the second the property of stochastic stability and its consequences. The third examines some perspectives in the light of some interesting recent development and provides some perspective.

2 Definitions

We consider a disordered model of Ising configurations $\sigma_n = \pm 1, n \in \Lambda \subset \mathcal{L}$ for some subset Λ (volume $|\Lambda|$) of some infinite graph $\mathcal{L}$. We denote by Σ_Λ the set of all $\sigma = \{\sigma_n\}_{n\in\Lambda}$, and $|\Sigma_\Lambda| = 2^{|\Lambda|}$. In the sequel the following definitions will be used.

1. *Hamiltonian*.
   For every $\Lambda \subset \mathcal{L}$ let $\{H_\Lambda(\sigma)\}_{\sigma\in\Sigma_N}$ be a family of $2^{|\Lambda|}$ *translation invariant* (*in distribution*) *Gaussian* random variables defined according to the general representation

$$H_\Lambda(\sigma) = -\sum_{X\subset\Lambda} J_X \sigma_X, \tag{2}$$

   where

$$\sigma_X = \prod_{i\in X} \sigma_i, \tag{3}$$

   ($\sigma_\emptyset = 0$) and the J's are independent Gaussian variables with mean

$$\mathrm{Av}(J_X) = 0, \tag{4}$$

   and variance

$$\mathrm{Av}(J_X^2) = \Delta_X^2. \tag{5}$$

2. *Average and covariance matrix*.
   The Hamiltonian $H_\Lambda(\sigma)$ has covariance matrix

$$\begin{aligned} \mathcal{C}_\Lambda(\sigma,\tau) &:= \mathrm{Av}(H_\Lambda(\sigma)H_\Lambda(\tau)) \\ &= \sum_{X\subset\Lambda} \Delta_X^2 \sigma_X \tau_X. \end{aligned} \tag{6}$$

   The two classical examples are the covariances of the Sherrington-Kirkpatrick model and the Edwards-Anderson model. A simple computation shows that the first is the square of the function (1) and the second is the link-overlap

$$\frac{1}{|\Lambda|} \sum_{|i-j|=1} \sigma_i \sigma_j \tau_i \tau_j. \tag{7}$$

By the Schwarz inequality

$$|\mathcal{C}_\Lambda(\sigma,\tau)| \le \sqrt{\mathcal{C}_\Lambda(\sigma,\sigma)}\sqrt{\mathcal{C}_\Lambda(\tau,\tau)} = \sum_{X\subset\Lambda} \Delta_X^2 \tag{8}$$

for all σ and τ.

3. *Thermodynamic stability.*
   The Hamiltonian (2) is thermodynamically stable if there exists a constant $\bar{c}$ such that

$$\sup_{\Lambda\subset\mathcal{L}} \frac{1}{|\Lambda|} \sum_{X\subset\Lambda} \Delta_X^2 \le \bar{c} < \infty. \tag{9}$$

   Thanks to the relation (8) a thermodynamically stable model fulfills the bound

$$\mathcal{C}_\Lambda(\sigma,\tau) \le \bar{c}|\Lambda| \tag{10}$$

   and has an order 1 normalized covariance

$$c_\Lambda(\sigma,\tau) := \frac{1}{|\Lambda|}\mathcal{C}_\Lambda(\sigma,\tau). \tag{11}$$

4. *Random partition function.*

$$\mathcal{Z}_\Lambda(\beta) := \sum_{\sigma\in\Sigma_\Lambda} e^{-\beta H_\Lambda(\sigma)}. \tag{12}$$

5. *Random free energy/pressure.*

$$-\beta\mathcal{F}_\Lambda(\beta) := \mathcal{P}_\Lambda(\beta) := \ln \mathcal{Z}_\Lambda(\beta). \tag{13}$$

6. *Random internal energy.*

$$\mathcal{U}_\Lambda(\beta) := \frac{\sum_{\sigma\in\Sigma_\Lambda} H_\Lambda(\sigma) e^{-\beta H_\Lambda(\sigma)}}{\sum_{\sigma\in\Sigma_\Lambda} e^{-\beta H_\Lambda(\sigma)}}. \tag{14}$$

7. *Quenched free energy/pressure.*

$$-\beta F_\Lambda(\beta) := P_\Lambda(\beta) := \mathrm{Av}(\mathcal{P}_\Lambda(\beta)). \tag{15}$$

8. *Random Boltzmann-Gibbs state*

$$\omega(-) := \sum_\sigma (-)\frac{e^{-\beta H_\Lambda}}{\mathcal{Z}_\Lambda(\beta)}, \tag{16}$$

   and its R-product version.

$$\Omega_\Lambda(-) := \sum_{\sigma^{(1)},\dots,\sigma^{(R)}} (-)\frac{e^{-\beta[H_\Lambda(\sigma^{(1)})+\cdots+H_\Lambda(\sigma^{(R)})]}}{[\mathcal{Z}_\Lambda(\beta)]^R}. \tag{17}$$

9. *Quenched overlap observables.*
For any smooth bounded function $G(c_\Lambda)$ (without loss of generality we consider $|G| \leq 1$ and no assumption of permutation invariance on G is made) of the covariance matrix entries we introduce (with a small abuse of notation) the random $R \times R$ matrix of elements $\{c_{k,l}\}$ (called *generalized overlap*) and its measure $\langle - \rangle_\Lambda$ by the formula

$$\langle G(c) \rangle_\Lambda := \mathrm{Av}(\Omega(G(c_\Lambda))). \tag{18}$$

E.g.: $G(c_\Lambda) = c_\Lambda(\sigma^{(1)}, \sigma^{(2)}) c_\Lambda(\sigma^{(2)}, \sigma^{(3)})$,

$$\langle c_{1,2} c_{2,3} \rangle_\Lambda = \mathrm{Av}\left(\sum_{\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}} c_\Lambda(\sigma^{(1)}, \sigma^{(2)}) c_\Lambda(\sigma^{(2)}, \sigma^{(3)}) \frac{e^{-\beta [\sum_{i=1}^{3} H_\Lambda(\sigma^{(i)})]}}{[\mathcal{Z}(\beta)]^3} \right). \tag{19}$$

3 Stochastic Stability

Given the Gaussian process $H_\Lambda(\sigma)$ of covariance $\mathcal{C}_\Lambda(\sigma, \tau)$ we introduce an independent Gaussian process, $K_\Lambda(\sigma)$, defined by the covariance $c_\Lambda(\sigma, \tau)$, the deformed random state

$$\omega_\Lambda^{(\lambda)}(-) = \frac{\omega(-e^{\lambda K_\Lambda})_\Lambda}{\omega(e^{\lambda K_\Lambda})_\Lambda} \tag{20}$$

and its relative deformed quenched state $\langle - \rangle_\Lambda^{(\lambda)} = \mathrm{Av}(\Omega_\Lambda^{(\lambda)}(-))$.

Definition 3.1 (Stochastic stability [1, 10]) A Gaussian spin glass model is stochastically stable if the deformed quenched state and the original one do coincide in the thermodynamic limit:

$$\lim_{\Lambda \nearrow \mathcal{L}} \langle - \rangle_\Lambda^{(\lambda)} = \lim_{\Lambda \nearrow \mathcal{L}} \langle - \rangle_\Lambda. \tag{21}$$

Remark 1 In spin glass models the existence of the thermodynamic limit has been settled only at the level of the free energy [11, 12]. For the correlation functions there are only abstract results using compactness arguments or, equivalently, existence along subsequences. In comparison with models without disorder, for instance the ferromagnetic ones, what is lacking is the control of the local correlation functions in terms of the interaction parameters. While ferromagnetic correlations increase monotonically when varying any spin interactions (Griffiths, Kelly and Sherman inequalities of type II [13, 14]) nothing is know about a spin glass correlation when the interaction distribution is changed. It is known nevertheless that spin glass correlation functions are not monotonic in the volume [15] when the interaction is centered. For non zero average of the interaction a special result exists in the Nishimori line where monotonicity is recovered [16].

Since the Hamiltonian H and the field K have a mutually rescaled distribution

$$H_\Lambda(\sigma) \stackrel{\mathcal{D}}{=} \sqrt{|\Lambda|} K_\Lambda(\sigma) \tag{22}$$

the addition law for the Gaussian variables implies

$$\sqrt{\beta^2 + \frac{\lambda^2}{|\Lambda|}} H(\sigma, J) \stackrel{\mathcal{D}}{=} \beta H(\sigma, J) + \lambda K(\sigma), \tag{23}$$

i.e. the deformation with a field K is equivalent to a change of the order $O(\frac{1}{N})$ in the temperature. The previous identity shows that the deformed measures do coincide, a part on points of discontinuity with respect to the temperature, with the original unperturbed one. Let consider some consequences of the stochastic stability that have been proved in a series of works.

Proposition 3.1 (Zero-average polynomials) *For every monomial Q of the overlap algebra (e.g. $c_{1,2}$, or $c_{1,2}^2 c_{2,3}$) for a Gaussian spin glass model defined by the covariance in* (6) *the following property hold*:

$$\left.\frac{\partial^2}{\partial\lambda^2}\langle Q\rangle_\Lambda^{(\lambda)}\right|_{\lambda=0} = \langle \Delta Q\rangle_\Lambda, \tag{24}$$

where the quantities ΔQ are polynomials which can be computed with a graph-theoretical algorithm or with the standard Parisi replica limit $n \to 0$ formula (*see* [1]). *Moreover the property* (23) *implies*

$$\frac{\partial}{2\beta\partial\beta}\langle Q\rangle_\Lambda = |\Lambda|\left.\frac{\partial^2}{\partial\lambda^2}\langle Q\rangle_\Lambda^{(\lambda)}\right|_{\lambda=0} = |\Lambda|\langle\Delta Q\rangle_\Lambda. \tag{25}$$

A simple computation allows to deduce that, for every interval $[\beta_0, \beta_1]$

$$\lim_{\Lambda\nearrow\mathcal{L}}\int_{\beta_0^2}^{\beta_1^2}\langle\Delta Q\rangle_\Lambda d\beta^2 = 0 \tag{26}$$

i.e. the vanishing in β^2-average of the quantities $\langle\Delta Q\rangle_\Lambda$ when the thermodynamic limit is considered. See also [17] *for an independent method to obtain the previous result which works for general* (*including non Gaussian*) *distributions and* [18] *for an interpretation of the identities as the Noether's conserved quantities in a classical mechanics theory.*

The consequences seen so far derive from the computation of the first two derivatives and basically mean that for a stochastic perturbation tuned by a parameter λ not only the first derivative vanishes when computed in zero (which is obvious for symmetry reasons) but also the second one in the thermodynamic limit does i.e. the curvature of the perturbed state is a vanishing function for increasing volumes.

The possibility that the computation of higher order derivatives could lead to new results beyond (26) has been investigated in [19, 20] and has a negative answer due to the following result

$$\left.\frac{\partial^{2n}}{\partial\lambda^{2n}}\langle Q\rangle_\Lambda^{(\lambda)}\right|_{\lambda=0} = (2n-1)!!\langle\Delta^n Q\rangle_\Lambda, \tag{27}$$

which implies that the vanishing of higher order derivatives provides the same information of the second one at the level of the whole algebra of observables.

In the paper [1] it has been proved that stochastic stability is equivalent to the following property:

Proposition 3.2 (Quenched additivity) *Given any finite collection of independent Gaussian fields $K_{(1)}(\sigma), K_{(2)}(\sigma), \dots, K_{(l)}(\sigma)$* (*independent also on the Hamiltonian*) *with the same covariance* (6), *and any smooth polynomially bounded functions $F_1, F_2, \dots, F_l$ a Gaussian* *spin glass model fulfills the following relation*

$$\lim_{\Lambda\nearrow\mathcal{L}} \mathrm{Av}\ln\Omega_\Lambda\left(\exp\left(\sum_{i=1}^{l} F_i(K_{(i)})\right)\right) = \lim_{\Lambda\nearrow\mathcal{L}} \sum_{i=1}^{l} \mathrm{Av}\ln\Omega_\Lambda(\exp F_i(K_{(i)})), \tag{28}$$

where the Gaussian measure $\mathrm{Av}(-)$ *include the integration on all the fields* $K_{(i)}$ *and the Hamiltonian, and the state* $\Omega(-)$ *is defined in* (17). *The previous formula would be trivial if the fields* F_i *would be independent with respect to the measure* Ω *which is not the case for the class of fields considered here. Equivalently, denoting the truncated expectations (cumulants) of order* p *of the Gibbs-Boltzmann state by* $\Omega(-;p)$, *the previous relation says that*

$$\lim_{\Lambda\nearrow\mathcal{L}} \mathrm{Av}\,\Omega_\Lambda\left(\sum_{i=1}^{l} F_i(K^{(i)});\, p\right) = \lim_{\Lambda\nearrow\mathcal{L}} \sum_{i=1}^{l} \mathrm{Av}\,\Omega_\Lambda\left(F_i(K^{(i)});\, p\right) \tag{29}$$

for every integer p, *i.e. the linearity in average of the truncated correlation functions of every order.*

4 Some Perspectives: Toward Ultrametricity

The method of stochastic stability is strictly related to the fluctuation method introduced in [21] and developed in [22]. The main idea of that method is to show that from the simple control of the fluctuations for the Hamiltonian per particle, which parallels the law of large numbers, one can deduce a set of identities. The relation between stochastic stability and the method of the fluctuations is developed in the paper [23] where it was proved that the linear part of the Ghirlanda-Guerra identities coincide with the identities produced by stochastic stability (see also [24] for a further interesting result in the framework of the competing particle systems).

The relation between stochastic stability and the method of fluctuation is still under investigation, in particular one would like to know if the two properties suffice to prove a much stronger property called ultrametricity: an overlap distribution is called ultrametric if it is supported (in the thermodynamic limit) only on isoceles and equilateral overlap configurations. For such distributions the measure of scalenes configurations is zero.

Some recent interesting developments are approching the solution of the problem. To explain them it is important to report a stronger definition of stochastic stability introduced in [1].

Definition 4.2 (Extended stochastic stability) A Gaussian spin glass model is stochastically stable in the extended sense if it is stochastically stable under deformation with respect to all $K^{(p)}_\Lambda(\sigma)$ whose covariance are $c^p_\Lambda(\sigma,\tau)$. Its consequences clearly extend to every power of the covariance c the results already proved with the standard stochastic stability.

In the framework of the competing particle systems it has been proved [25] that the extended stochastic stability if applied to overlaps which take only a finte number of values implies ultrametricity. The same result has been obtained, for the same class of overlap distribution, using an extended version of the fluctuation method in [26, 27].

It is important to stress that the formalism developed so far is not limited to the mean field spin glasses. Indeed all the results which we have shortly reviewed hold for a general

Gaussian spin glass model in terms of the proper covariance matrix. In particular the recent developments, in which ultrametricity under certain hypotheses has been proved, do not distinguish between the mean field case or the short-range finite-dimensional one like the Edwards-Anderson model. They give indeed a clear indication that the factorization structure of the two cases is likely to be the same. Of course even such similarity for the factorization structure wouldn't be enough to guarantee the same low-temperature phase for the two models. The celebrated Parisi self-consistence equation [28], which implies that the overlap distribution of the Sherrington-Kirlpatrick model has a non-trivial support, is in fact very specific for the mean field case with its strong permutation invariance symmetry. It is still not clear what would be the structure of the link-overlap distribution for the Edwards-Anderson model if a complete ultrametric factorization would take place.

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